

KINETIC AND STATISTICAL METHODS IN TRANSPORT THEORY

MACROSCOPIC DESCRIPTION OF THE DYNAMICS OF DISTURBANCES OF A WEAKLY IONIZED PLASMA UNDER HIGH-NONEQUILIBRIUM CONDITIONS

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Consideration is given to the dynamics of disturbances of a charged weakly ionized plasma on exposure to a fairly strong electric field in the presence of high velocities and disturbance gradients. Equations for macroscopic description of the plasma are constructed by the projective-kinetic method; the general expressions for correlation functions and their related nonequilibrium kinetic coefficients are obtained. With the example of the correlation density function and the diffusion coefficient of a highly nonequilibrium plasma, the asymptotics of its kinetic coefficients is considered.

Keywords: weakly ionized plasma, macroscopic description, disturbance dynamics, strong electric fields, high-nonequilibrium states.

Introduction. The work is a continuation of the earlier investigations [1, 2] of the dynamics of disturbances of a weakly-ionized-plasma subsystem in external electric fields. Problems of this kind arise in studying weakly-ionized plasma flows; their solutions are well known in the case of weak and moderate fields [3, 4]. The plasma's high-nonequilibrium states due to the large electric fields, particles velocities, and disturbance gradients, conversely, are still not clearly understood and need additional investigations.

Initial Equations. Formulation of the Problem. For construction of a closed macroscopic description of disturbances of the plasma subsystem we use a projective-kinetic method proposed in [5, 6]; according to this method, if a disturbance of the system under study is microscopically characterized by the distribution function δf satisfying a kinetic equation (of the Boltzmann equation type):

$$(\partial_t - E) \varphi_s = 0, \quad \delta f = \varphi_s^f \equiv \varphi_s, \quad (1)$$

and is macroscopically characterized by the set of determining macroscopic quantities (DMQs)

$$a_\alpha \equiv (\alpha, \varphi_s) \equiv \int_\Omega \alpha \delta f d\Omega, \quad (2)$$

then, separating, with the projection operator \hat{p} , the disturbance of the distribution function δf and the evolution operator E into a collinear part and that (corresponding to irregular fast-relaxing motions) orthogonal to the DMQ space:

$$\varphi_s = \hat{p}\varphi_s + \hat{\varphi}_s, \quad E = \hat{p}E + \hat{E}, \quad \hat{p}\varphi_s \equiv \alpha_s (\alpha\varphi_s), \quad \hat{\varphi}_s, \hat{E}, \hat{R} \equiv (1 - \hat{p}) \varphi_s, E, R,$$

we obtain from Eq. (1), on integration with microattributes and a number of transformations, a system of equations for DMQs and closing relations:

$$[\delta_{\alpha\beta} \partial_t - (\alpha, E\beta_s)] a_\beta = (\alpha, E\hat{\varphi}_s), \quad (\alpha, E\hat{\varphi}_s) = (\alpha, E\hat{R}\hat{E}\beta_s) a_\beta, \quad (3)$$

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where $R \equiv (\partial_t - E)^{-1}$.

For translation-invariant systems in the Fourier–Laplace transformation Eqs. (3) take an algebraic form and can be expressed by the correlation functions

$$[A^{\beta\alpha} / \det(R_{\alpha\beta})] a_\beta = a_{\alpha 0}, \quad R_{\alpha\beta} \equiv (\alpha, R\beta_s), \quad (4)$$

where $A^{\beta\alpha}$ is the algebraic complement of the element in the determinant $\det(R_{\alpha\beta})$.

Consideration is given to the electron subsystem of a weakly ionized plasma in an equilibrium thermostat where a fairly strong (preconduction) electric field has been produced. In this case the prevailing mechanism of scattering for gas plasmas is the carrier scattering on the atoms of a neutral component. As a consequence of the low density and the strong longitudinal field, a one-dimensional (as far as disturbances are concerned) problem is investigated, whereas the influence of induced magnetic fields is disregarded.

Kinetic and Macroscopic Equations of Disturbances Dynamics. The kinetic equation for the single-particle distribution function has the form

$$[\partial_t + \mathbf{V} \cdot \nabla + \mathbf{F} \cdot \partial_p + I] f = (\partial_t - E) f = 0, \quad (5)$$

where $\mathbf{F} = q(E_s + \mathbf{E}_{\text{ind}})$ and $\nabla \cdot \mathbf{E}_{\text{ind}} = q\delta n / \epsilon_0$; as the collision integral I we use the first approximation of a generalized v moment model

$$If = -v(V) [f - f_0 \xi_n K_{nm}(\chi_m, vf)] \Rightarrow -v(V) [f_s - n_v f_0]. \quad (6)$$

Here K_{nm} are the matrix elements of the kernel of the integral collision operator represented in the form of the expansion in a complete bilinear system of functions $(\zeta\chi)$ ($\zeta_1 = \chi_1 = 1$ and $K_{11} = 1$), $v(V)$ is the frequency of collisions of the subsystem's particles with the neutral-component particles (it is generally a function of the particle velocity: $n_v = (1, vf_s) / (1, vf_0)$), and f_s and f_0 are the steady-state and equilibrium solutions of Eq. (5). On passage to a dimensionless form in the Fourier–Laplace transformation ($r, t \rightarrow k, z$) and the approximation in question ($v = \text{const} = 1$), for the steady-state distribution and its disturbance we obtain, from (5),

$$f_s = \frac{1}{c\pi^{3/2}} \exp(-V_\perp^2) \int_{-\infty}^{V_\parallel} \exp(-u^2 + (u - V_\parallel)/c) du, \quad (7)$$

$$\varphi_s = \frac{\delta n}{c} \int_{-\infty}^{V_\parallel} \exp(\Lambda) [f_0 - \chi^* (f_s - f_0)] du + \varphi_0, \quad \delta n = (1, \varphi_s),$$

where

$$\Lambda = -\frac{1}{c} \int_u^{V_\parallel} [z +iky + v] dy; \quad f_0 = \frac{1}{\pi^{3/2}} \exp[-(V_\parallel^2 + V_\perp^2)];$$

$$c = \frac{qE_s}{mV_{\text{th}0}^2 v_0}; \quad \chi^* = \frac{ik}{ck^2} \chi; \quad \chi = \frac{4\pi^2 n_0}{m\epsilon_0 v_0^2} = \frac{\omega_q^2}{v_0^2}.$$

The form of the steady-state distribution for a number of the characteristic mechanisms of scattering and an analysis of the spectrum of disturbances of the subsystem in question have been given in [1, 2]. For macroscopic description of the subsystem's disturbance, we select, as the GMQs, the disturbances of the particle number density δn

= (Ψ_1, φ_s) , the momentum density $\delta p = (\Psi_2, \varphi_s)$, and the density of the subsystem's particle energy $\delta e = (\Psi_3, \varphi_s)$ whose microattributes ($\alpha \equiv \Psi_\alpha$) are respectively equal to

$$\Psi_1 = 1, \quad \Psi_2 = n_2 (V - V_s), \quad \Psi_3 = n_3 [(V - V_s)^2 + \alpha_2 (V - V_s) + \alpha_3],$$

where n_i and α_i are the normalization constants determined by the conditions

$$(\Psi_\alpha, \Psi_\beta, f_s) \equiv (\alpha, \beta_s) = \delta_{\alpha\beta}, \quad V_s = (V_1 f_s) = c.$$

The system of equations (3) in this case takes the form

$$\begin{aligned} (z + ikc) \delta n + ikW_{12} \delta p &= 0, \\ (ikW_{21} + d_{21}) \delta n + (z + ikW_{22} + \Lambda_{22}) \delta p + (ikW_{23} + d_{23}) \delta e &= \delta p_0, \\ (ikW_{31} + d_{31}) \delta n + (ikW_{32} + cF_{32} + d_{32}) \delta p + (z + ikW_{33} + \Lambda_{33}) \delta e &= \delta e_0. \end{aligned} \tag{8}$$

Here $W_{\alpha\beta} = (\alpha, V\beta_s)$, $F_{\alpha\beta} = (\alpha, F_\gamma \partial_{p\gamma} \beta_s)$ and $\nu_{\alpha\beta} = (\alpha, I\beta_s)$ are the gradient, force, and relaxation contributions of the evolution operator respectively.

$$\begin{aligned} W_{12} &= n_2^{-1}, \quad W_{21} = W_{12} + n_2 \frac{\chi}{k^2}, \quad W_{22} = c(1 + 2c^2 n_2^2), \\ W_{31} &= \alpha_2 n_3 \frac{\chi}{k^2}, \quad W_{23} = W_{32} = \frac{n_2(1 - n_3^2)}{n_3}, \quad F_{32} = -\frac{2n_3}{n_2}, \\ W_{33} &= c \left\{ 1 + n_3^2 \left[44c^4 + 14c^2 + \frac{15}{4} + 8n_2^4 c^8 - 2c^2 n_2^2 \left(2c^4 + 4c^2 + \frac{5}{4} \right) \right] \right\}, \\ \Lambda_{\alpha\beta} &= \nu_{\alpha\beta} + d_{\alpha\beta}, \quad \nu_{\alpha\beta} = \nu_{\alpha\alpha} \delta_{\alpha\beta}, \quad d_{\alpha\beta} = -\frac{(\gamma E, R\hat{E}\beta_s)}{z \det(R_{\alpha\beta})} A^{\gamma\alpha}. \end{aligned}$$

In Eqs. (8), allowance has also been made for the steady-state relations: $cF_{\alpha\beta} = -\nu_{\alpha\beta}$. For the exact values of the coefficients $d_{\alpha\beta}$, system (8) becomes equivalent to the initial kinetic equations for the average distribution-function moments. Generally, $d_{\alpha\beta}$ are quantities of higher order, as far as the nonequilibrium parameters (gradients, fields, and velocities) are concerned, than the remaining coefficients of the system. Therefore, by analogy with nonequilibrium thermodynamics with both low and high nonequilibriums, we can represent them as an expansion in these parameters, thus performing a successive approximate analysis. In our case it is sufficient to allow for them only as for diagonal corrections to frequencies.

Kinetic Coefficients and Correlation Functions. For obtaining the general expressions of nonequilibrium kinetic coefficients and for analysis of their behavior under high-nonequilibrium conditions, it is expedient to represent the system of macroscopic equations by correlation functions, which are the observed quantities. For correlation functions, with account for the explicit form of the operator R and the solutions (7), there follows the general equation

$$R_{nm} = Q_{nm}^s + [Q_{n0}^0 - \chi^* (Q_{n0}^s - Q_{n0}^0)] Q_{0m}^s / D, \tag{9}$$

where $D = 1 - Q_{00}^0 + \chi^* (Q_{00}^s - Q_{00}^0)$.

The special functions

$$Q_{nm}^{s0} \equiv \frac{1}{c} \int_{-\infty}^{+\infty} dV_{\parallel} dV_{\perp} V_{\perp}^n \int du u^m \exp(\Lambda(u, V_{\parallel}, V_{\perp})) f_{s0}(u, V_{\perp}) \quad (10)$$

using the recurrence relations

$$Q_{mn}^{s0} = \frac{1}{2\alpha} \left[(m-1) Q_{m-2,n}^{s0} - \tilde{z} Q_{m-1,n}^{s0} + \frac{1}{c} (V_{\parallel}^{m+n-1}, f_{s0}) \right],$$

$$Q_{mm}^{s0} = \lambda^{s0} \left[\frac{1}{c} (V_{\parallel}^{m+n-1}, f_{s0}) - (n-1) Q_{m,n-2}^{s0} - \tilde{z} Q_{m,n-1}^{s0} + \frac{1}{c} (Q_{m,n-1}^{s0} - Q_{m,n-1}^0) \right],$$

$$\alpha \equiv \frac{ik}{2c}, \quad \tilde{z} \equiv \frac{z+1}{c}, \quad \lambda^s \equiv \frac{1}{2\alpha}, \quad \lambda^0 \equiv \frac{1}{2(\alpha-1)}$$

are reduced to the functions $Q_{00}^{s,0}$ for which the asymptotics are found for large k , z , and c :

$$k \gg 1; \quad z, c \approx 1, \quad Q_{00}^{(0)} \Rightarrow \sqrt{\frac{\pi}{k^2}}, \quad Q_{00}^{(s)} \Rightarrow \xi(c) Q_{00}^{(0)}, \quad \xi(c) = \frac{\sqrt{\pi}}{2c} \exp(1/4c^2) \operatorname{erfc}\left(\frac{1}{2c}\right);$$

$$c \gg 1; \quad k, z \approx 1, \quad Q_{00}^{(0)} \Rightarrow \sqrt{\frac{\pi}{2ikc}} = \pm \frac{1}{2} \sqrt{\frac{\pi}{kc}} (1-i), \quad Q_{00}^{(s)} \Rightarrow \frac{\pi}{2ikc};$$

$$z \gg 1; \quad k, c \approx 1, \quad Q_{00}^{(0)} \Rightarrow Q_{00}^{(s)} \Rightarrow \frac{1}{z}.$$

Using the last expressions we can obtain any correlation functions in asymptotic form. For example, for the correlation density function (CDF), we have

$$\text{CDF} \equiv R_{11} = \frac{Q_{00}^{(s)}}{D}; \quad R_{11}(k \gg 1) \Rightarrow \frac{\sqrt{\pi}}{|k|} \xi(c); \quad R_{11}(z \gg 1) \Rightarrow \frac{1}{z}; \quad R_{11}(c \gg 1) \Rightarrow \frac{\pi}{2kc}. \quad (11)$$

The correlation density function near the point of exchange of stabilities ($D \rightarrow 0$) is described by the unipolar approximation $\operatorname{Re}(R_{11}) = A(z)/(z - z_{cr})$ and experiences an abnormal stepwise change as it passes this point.

Let us consider the procedure of obtaining the expressions of kinetic coefficients using the nonequilibrium diffusion coefficient (NDC) as an example. For this purpose we need only a system of two GMQs: the disturbance of the particle number density and that of the momentum density (in dimensionless variables, the disturbance of momentum is equal to the disturbance of velocity and the disturbance of the electric-current density, i.e., $\delta p = \delta V = \delta j$). In this case the system of equations (4) and its coefficients take the form

$$\frac{R_{22}}{B} \delta n + \frac{R_{12}}{B} \delta j = 0;$$

$$-\frac{R_{21}}{B} \delta n + \frac{R_{11}}{B} \delta j = 0;$$

$$B = R_{11}R_{22} - R_{12}R_{21}; \quad R_{11} = Q_{00}^s/D; \quad R_{12} = n_2(Q_{01}^s - cQ_{00}^s)/D;$$

$$R_{21} = \frac{n_2}{D} \left\{ Q_{10}^s D - cQ_{00}^s + Q_{00}^s [Q_{10}^0 - \chi^* (Q_{10}^s - Q_{10}^0)] \right\};$$

$$R_{22} = \frac{n_2^2}{D} \left\{ D (Q_{11}^s - cQ_{10}^s) + (Q_{01}^s - cQ_{00}^s) [Q_{10}^0 - \chi^* (Q_{10}^s - Q_{10}^0) - c] \right\}. \quad (12)$$

The second equation of system (12) yields for δj :

$$\delta j = \frac{R_{21}}{R_{11}} \delta n = -ik (\text{NDC})_{\text{ef}} \delta n = -ikd\delta n + \mu\delta E, \quad (13)$$

where $(\text{NDC})_{\text{ef}}$ is the effective nonequilibrium diffusion coefficient in which, taking into account the explicit form of the correlators, we can separate the purely diffusion and field contributions: $\delta E = -ik\frac{\chi}{k^2}\delta n$ and $\mu = \frac{\delta j}{\delta E}$; δE is the induced field and μ is the differential conductivity.

From Eqs. (12) and (13), on reduction of the functions $Q_{nm}^{s,0}$ to the functions $Q_{00}^{s,0}$ and a number of transformations, for the Fourier–Laplace transforms of the diffusion coefficient and the differential conductivity we obtain

$$d = \frac{n_2}{k^2 Q_{00}^s} [- (z + ikc) Q_{00}^s + (1 - Q_{00}^0)]; \quad \mu = \frac{ikn_2}{ck^2 Q_{00}^s} (Q_{00}^s - Q_{00}^0). \quad (14)$$

Expressions (14) yield, with asymptotics of the functions $Q_{00}^{s,0}$, the asymptotic expressions of the diffusion coefficient and the differential conductivity within large velocities, gradients, and fields:

$$\begin{aligned} z \gg 1, \quad k, c \sim 1: \quad d &\Rightarrow \frac{n_2}{z} \left[(c^2 + 0.5) + \frac{2ikc}{k^2} \right], \quad \mu \Rightarrow \frac{n_2}{z}; \\ k \gg 1, \quad z, c \sim 1: \quad d &\Rightarrow \frac{n_2}{\sqrt{\pi} |k|} \left[\frac{1 - ic\sqrt{\pi} \xi(c)}{\xi(c)} \right], \quad \mu \Rightarrow \frac{in_2}{c|k|} \left(1 - \frac{1}{\xi(c)} \right); \\ c \gg 1, \quad z, k \sim 1: \quad d &\Rightarrow \frac{1}{\pi |k|} (2 - i\pi), \quad \mu \Rightarrow \frac{ik}{ck^2} \left[1 \pm \sqrt{\frac{kc}{\pi}} (1 - i) \right]; \end{aligned} \quad (15)$$

$$n_2 = (c^2 + 0.5)^{-1/2}.$$

It follows from (15) that at high frequencies, the subsystem displays a behavior characteristic of solid bodies, and when the fields and inhomogeneities are large the differential conductivity can change its sign, becoming negative. The change of sign by μ for large fields is related to the condition $\sqrt{kc} \geq 1$; this corresponds to a possible instability with an increment $\approx \sqrt{kc}$ in the subsystem [1, 7]. This result is a nonequilibrium generalization of the well-known criterion of negative differential conductivity in plasma-stability theory where it has been linked only to the mechanisms of scattering [3]. It follows from (15) that the criterion of negative differential conductivity is of a more universal character due to the high nonequilibrium and is not necessarily related to the mechanisms of scattering.

To allow for other effects and to obtain the remaining kinetic coefficients we should extend the set of DMQs. For example, on introduction of another DMQ — energy density — the equation for the current disturbance takes the form

$$\delta j = -ikd_1 \delta n - ikd_2 \delta e, \quad (16)$$

$$d_1 = -\frac{ik R_{23}R_{31} - R_{21}R_{33}}{k^2 R_{11}R_{33} - R_{31}R_{13}}; \quad d_2 = -\frac{ik R_{21}R_{13} - R_{11}R_{23}}{k^2 R_{11}R_{33} - R_{31}R_{13}},$$

where d_1 is the effective nonequilibrium diffusion coefficient and d_2 is the effective nonequilibrium thermodiffusion coefficient; for these coefficients, as above, we can obtain specific expressions by the functions $Q_{nm}^{s,0}$ and perform an asymptotic analysis.

Conclusions. On the basis of the projective-kinetic method, we have performed the closed macroscopic description of the dynamics of disturbances of a charged system in a strong electric field. We have obtained the equations for the particle number density, the momentum density, and the energy density, have given the general expressions for correlation functions, and have indicated the method of finding nonequilibrium kinetic coefficients. For the correlation density function and the effective diffusion coefficient, we have obtained the asymptotic expressions within large fields, gradients, and velocities. It has been shown that at large frequencies, the plasma displays the characteristic solid-state properties, and when the fields and gradients are large the differential conductivity (irrespective of the mechanism of scattering) changes its sign, becoming negative. The latter extends the criterion of negative differential conductivity, which is well known in plasma-stability theory, to high-nonequilibrium states.

NOTATION

$A(z)$, denominator in the polar expression of the correlation density function; $A^{\beta\alpha}$, algebraic complement; $a_\alpha = (\alpha, \varphi_s)$, determining macroscopic quantity; $a_{\alpha 0}$, initial value of a_α ; $\alpha \equiv ik/2c$; α_i , normalization constant; c , ratio of the drift velocity of the subsystem's particles to the thermal velocity; D , spectral function; d , diffusion coefficient; $d_{\alpha\beta}$, kinetic coefficient; $\det(R_{\alpha\beta})$, determinant with elements $R_{\alpha\beta}$; e , energy of the subsystem's particles; E , evolution operator of the system; \mathbf{E}_s and \mathbf{E}_{ind} , strengths of the external steady-state and induced electric fields; f , f_s , and f_0 , full, steady-state, and equilibrium distribution functions of the subsystem's particles; $F_{\alpha\beta}$, coefficients of dynamic equations; I , collision integral; i , imaginary unit; j , electric-current density; k , disturbance gradient in the Fourier–Laplace transformation (transformation parameter); m , mass in formulas (7); n , concentration; n_0 , equilibrium value of the particle concentration; n_i , normalization constants; p , momentum; \hat{p} , projection operator; q , charge of the subsystem's particles; r , space coordinate; $Q_{nm}^{s,0}$, special function; R , resolvent operator; $R_{\alpha\beta}$, correlation function; Re , real part of the functions; t , time variable; u , integration variable; V , dimensionless velocity of the subsystem's particles; V_{\parallel} and V_{\perp} , longitudinal and transverse (with respect to the field) velocity components; $V_{\text{th}0}$, equilibrium thermal velocity of the subsystem's particles; $W_{\alpha\beta}$, coefficients of dynamic equations; z , rate of change in the disturbances in the Fourier–Laplace transformation (transformation parameter); z_{cr} , unstable root of the spectral equation; α , β , γ , and Ψ_α , microattributes of the corresponding DMQs; δ , index of deviation of the quantities from steady-state values; $\delta_{\alpha\beta}$, Kronecker symbol; $\delta f \equiv \varphi_s$, disturbance of the distribution function; ξ_i and χ_i , complete system of polynomial velocity functions; $\lambda^{s,e}$, Λ , $\Lambda_{\alpha\beta}$, and ξ , conventional symbols of different functions; ν , collision frequency of the subsystem's particles; ν_0 , equilibrium value of the collision frequency; χ , dimensionless plasma-oscillation frequency squared; φ_0 , initial value of the disturbance of the distribution function; ω_q , frequency of carrier plasma oscillations; $\partial_\beta \equiv \frac{\partial}{\partial \beta}$, $\partial_t \equiv \frac{\partial}{\partial t}$, $\nabla \equiv i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$, and $\partial_p \equiv i\frac{\partial}{\partial p_x} + j\frac{\partial}{\partial p_y} + k\frac{\partial}{\partial p_z}$, symbols of the derivatives; $(\alpha, \beta_s) \equiv (\alpha, \beta f_s) \equiv \int_{\Omega} \alpha(\Omega)\beta(\Omega)\delta f(\Omega)d\Omega$, phase-space average (Ω , phase-space variables). Subscripts and superscripts: s, steady-state; 0, equilibrium; ind, induced; th, thermal; cr, critical; ef, effective.

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